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STABILITY OF A RECTANGULAR PLATE UNDER BIAXIAL TENSION

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The stability problem of a rectangular plate undergoing uniform biaxial in-plane tensile strain is solved using the three-dimensional equations of nonlinear elasticity. The surfaces of the plate are stress-free, and special boundary conditions that allow one to separate variables in the linearized equilibrium equations are specified on the lateral surfaces. For three particular models of incompressible materials, the critical curves are constructed and the instability region is determined in the plane of the loading parameters (the multiplicities of elongations of the plate material in the unperturbed equilibrium state). The numerical results show that for thin plates loaded by tensile stresses, the size and shape of the instability region depend only slightly on the relation among the length, width, and thickness of the plate. Based on the results obtained, a simple approximate stability criterion is proposed for an elastic plate under tensile loads.

Key words: *nonlinear elasticity, stability of deformable bodies.*

1. Unperturbed Equilibrium State. We consider the uniform strain [1]

$$X_1 = \lambda_1 x_1, \qquad X_2 = \lambda_2 x_2, \qquad X_3 = \lambda_3 x_3,$$
$$\lambda_1 = \text{const}, \qquad \lambda_2 = \text{const}, \qquad \lambda_3 = \text{const}, \tag{1.1}$$
$$0 \leqslant x_1 \leqslant a, \qquad 0 \leqslant x_2 \leqslant b, \qquad |x_3| \leqslant h/2$$

of an elastic rectangular plate loaded at the edges $x_1 = 0, a$ and $x_2 = 0, b$ by distributed normal forces of intensities q_1 and q_2, respectively. In relations (1.1), x_1, x_2, and x_3 are the Cartesian coordinates in the undeformed state (Lagrangian coordinates), X_1, X_2, and X_3 are the Eulerian Cartesian coordinates, λ_1, λ_2, and λ_3 are the multiplicities of the elongations along the coordinate axes, and a, b, and h are the side lengths and thickness of the undeformed plate, respectively. The strain gradient C is given by

$$C = \lambda_1 \boldsymbol{i}_1 \boldsymbol{i}_1 + \lambda_2 \boldsymbol{i}_2 \boldsymbol{i}_2 + \lambda_3 \boldsymbol{i}_3 \boldsymbol{i}_3. \tag{1.2}$$

Here $\boldsymbol{i}_1$, $\boldsymbol{i}_2$, and $\boldsymbol{i}_3$ is the orthonormal vector basis of the Cartesian coordinates. The incompressibility condition $\det C = 1$ implies the expression for the elongation multiplicity λ_3 in terms of the strain parameters λ_1 and λ_2: $\lambda_3 = (\lambda_1 \lambda_2)^{-1}$.

The expression for the Cauchy–Green strain measure G corresponding to (1.1) is given by

$$G = C \cdot C^{\rm t} = \sum_{k=1}^{3} G_k \boldsymbol{i}_k \boldsymbol{i}_k, \qquad G_k = \lambda_k^2. \tag{1.3}$$

The Piola stress tensor D is written as

$$D = P \cdot C, \tag{1.4}$$

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where P is the Kirchhoff stress tensor, which is given by the formula [1]

$$P = 2W(G)_{,G} - \sigma G^{-1}$$

for an incompressible isotropic material and is equal to

$$P = \sum_{k=1}^{3} P_k \boldsymbol{i}_k \boldsymbol{i}_k = (\chi_1 - \sigma\lambda_1^{-2})\boldsymbol{i}_1\boldsymbol{i}_1 + (\chi_2 - \sigma\lambda_2^{-2})\boldsymbol{i}_2\boldsymbol{i}_2 + (\chi_3 - \sigma\lambda_3^{-2})\boldsymbol{i}_3\boldsymbol{i}_3,$$

$$\chi_k = 2\,\frac{\partial W(G_1, G_2, G_3)}{\partial G_k}, \qquad k = 1, 2, 3 \tag{1.5}$$

for biaxial tension of the plate (1.1). Here $W(G_1, G_2, G_3)$ is the specific potential energy of the elastic plate. For the unperturbed state, the pressure σ is determined from the condition of the stress-free surfaces of the plate, and with allowance for (1.5), it is given by

$$\sigma = \lambda_3^2 \chi_3. \tag{1.6}$$

Using (1.2) and (1.5), from (1.4) we determine the intensities q_1 and q_2 of the normal forces per unit area in the undeformed configuration:

$$q_1 = \boldsymbol{i}_1 \cdot D \cdot \boldsymbol{i}_1|_{x_1=0,a} = \lambda_1\chi_1 - \frac{\lambda_3^2}{\lambda_1}\,\chi_3, \qquad q_2 = \boldsymbol{i}_2 \cdot D \cdot \boldsymbol{i}_2|_{x_2=0,b} = \lambda_2\chi_2 - \frac{\lambda_3^2}{\lambda_1}\,\chi_3.$$

One can easily show that the following energy relations are valid for q_1 and q_2:

$$q_1 = \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1}, \qquad q_2 = \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_2}. \tag{1.7}$$

2. Linearized Equilibrium Equations. We consider a small perturbation of the equilibrium state considered above. The linearized equilibrium equations of an incompressible body [1, 2] which describe this perturbed state are written in terms of the undeformed configuration as

$$\mathring{\nabla} \cdot \Theta = 0, \qquad \mathring{\nabla} = \boldsymbol{i}_1\,\frac{\partial}{\partial x_1} + \boldsymbol{i}_2\,\frac{\partial}{\partial x_2} + \boldsymbol{i}_3\,\frac{\partial}{\partial x_3}; \tag{2.1}$$

$$(C^{-1} \cdot \mathring{\nabla}) \cdot \boldsymbol{w} = 0, \qquad \boldsymbol{w} = w_1\boldsymbol{i}_1 + w_2\boldsymbol{i}_2 + w_3\boldsymbol{i}_3; \tag{2.2}$$

$$\Theta = D^* = P^* \cdot C + P \cdot C^*, \qquad C^* = \mathring{\nabla}\boldsymbol{w}, \qquad D^* = \Big(\frac{d}{d\eta}\,D(\boldsymbol{R} + \eta\boldsymbol{w})\Big)_{\eta=0}. \tag{2.3}$$

Here Θ is the linearized Piola stress tensor, $\mathring{\nabla}$ is the nabla operator for the undeformed plate configuration, $\boldsymbol{R}$ is the radius vector in the unperturbed state, and $\boldsymbol{w}$ is the vector of the additional displacements.

At the surfaces of the plate $x_3 = \pm h/2$, the boundary conditions

$$\boldsymbol{i}_3 \cdot \Theta|_{x_3=\pm h/2} = 0 \tag{2.4}$$

imply the absence of loads in the perturbed state.

To find an expression for P^* in (2.3), we use the fact that for an isotropic body, the tensors P and G are coaxial, i.e., their eigenvectors $\boldsymbol{e}_k$ $(k = 1, 2, 3)$ coincide. In the unperturbed state, the unit vectors $\boldsymbol{e}_k$ and $\boldsymbol{i}_k$ coincide. From (1.3) and (1.5), it follows that

$$P^* = \sum_{k=1}^{3}(P_k^*\boldsymbol{e}_k\boldsymbol{e}_k + P_k\boldsymbol{e}_k^*\boldsymbol{e}_k + P_k\boldsymbol{e}_k\boldsymbol{e}_k^*); \tag{2.5}$$

$$G^* = \sum_{k=1}^{3}(G_k^*\boldsymbol{e}_k\boldsymbol{e}_k + G_k\boldsymbol{e}_k^*\boldsymbol{e}_k + G_k\boldsymbol{e}_k\boldsymbol{e}_k^*). \tag{2.6}$$

Since the vectors $\boldsymbol{e}_k$ and $\boldsymbol{e}_k^*$ $(k = 1, 2, 3)$ are mutually orthogonal $(\boldsymbol{e}_k \cdot \boldsymbol{e}_k^* = 0)$, from (2.5) and (2.6) we obtain

$$\boldsymbol{e}_k \cdot P^* \cdot \boldsymbol{e}_k = P_k^*; \tag{2.7}$$

$$\boldsymbol{e}_k \cdot G^* \cdot \boldsymbol{e}_k = G_k^*; \tag{2.8}$$

$$\boldsymbol{e}_i \cdot P^* \cdot \boldsymbol{e}_j = \frac{P_i - P_j}{G_i - G_j}\, \boldsymbol{e}_i \cdot G^* \cdot \boldsymbol{e}_j. \tag{2.9}$$

Here $i, j = 1,\ 2,\ 3$ $(i \neq j)$.

The relations for P_k^* are obtained from (1.5):

$$P_k^* = \sum_{s=1}^{3} \chi_{ks} G_s^* + qG_k^{-1} + \sigma G_k^{-2} G_k^*, \qquad q = -\sigma^*, \qquad \chi_{ks} = \frac{\partial \chi_k(G_1, G_2, G_3)}{\partial G_s}, \qquad k, s = 1, 2, 3. \tag{2.10}$$

Formulas (2.7) and (2.9) give representations of all components of the tensor P^* in the basis $\boldsymbol{e}_k$ in terms of the components of G^*. The quantities G_k^* in (2.10) are determined according to (2.8), and the tensor G^* is given by

$$G^* = C \cdot \overset{\circ}{\nabla} \boldsymbol{w}^{\mathrm{t}} + \overset{\circ}{\nabla} \boldsymbol{w} \cdot C^{\mathrm{t}} = \sum_{k,s=1}^{3} \Big(\lambda_k \frac{\partial w_k}{\partial x_s} + \lambda_s \frac{\partial w_s}{\partial x_k} \Big) \boldsymbol{i}_k \boldsymbol{i}_s. \tag{2.11}$$

Using (1.2), (1.5), and (2.7)–(2.11), from (2.3) we find the components of the linearized Piola stress tensor Θ in the basis $\boldsymbol{i}_k$:

$$\Theta_{11} = \Big(\chi_1 + \frac{\sigma}{\lambda_1^2} \Big) \frac{\partial w_1}{\partial x_1} + \lambda_1 \chi_1^* + \frac{q}{\lambda_1}, \qquad \Theta_{13} = \frac{\lambda_1^2 \chi_1 - \sigma}{\lambda_1^2 - \lambda_3^2} \Big(\frac{\partial w_3}{\partial x_1} + \frac{\lambda_3}{\lambda_1} \frac{\partial w_1}{\partial x_3} \Big),$$

$$\Theta_{12} = \Big(\frac{\lambda_1 \lambda_2 (\chi_1 - \chi_2)}{\lambda_1^2 - \lambda_2^2} + \frac{\sigma}{\lambda_1 \lambda_2} \Big) \frac{\partial w_1}{\partial x_2} + \frac{\lambda_1^2 \chi_1 - \lambda_2^2 \chi_2}{\lambda_1^2 - \lambda_2^2} \frac{\partial w_2}{\partial x_1},$$

$$\Theta_{21} = \Big(\frac{\lambda_1 \lambda_2 (\chi_1 - \chi_2)}{\lambda_1^2 - \lambda_2^2} + \frac{\sigma}{\lambda_1 \lambda_2} \Big) \frac{\partial w_2}{\partial x_1} + \frac{\lambda_1^2 \chi_1 - \lambda_2^2 \chi_2}{\lambda_1^2 - \lambda_2^2} \frac{\partial w_1}{\partial x_2},$$

$$\Theta_{22} = \Big(\chi_2 + \frac{\sigma}{\lambda_2^2} \Big) \frac{\partial w_2}{\partial x_2} + \lambda_2 \chi_2^* + \frac{q}{\lambda_2}, \qquad \Theta_{23} = \frac{\lambda_2^2 \chi_2 - \sigma}{\lambda_2^2 - \lambda_3^2} \Big(\frac{\partial w_3}{\partial x_2} + \frac{\lambda_3}{\lambda_2} \frac{\partial w_2}{\partial x_3} \Big),$$

$$\Theta_{31} = \frac{\lambda_1^2 \chi_1 - \sigma}{\lambda_1^2 - \lambda_3^2} \Big(\frac{\partial w_1}{\partial x_3} + \frac{\lambda_3}{\lambda_1} \frac{\partial w_3}{\partial x_1} \Big), \qquad \Theta_{32} = \frac{\lambda_2^2 \chi_2 - \sigma}{\lambda_2^2 - \lambda_3^2} \Big(\frac{\partial w_2}{\partial x_3} + \frac{\lambda_3}{\lambda_2} \frac{\partial w_3}{\partial x_2} \Big), \tag{2.12}$$

$$\Theta_{33} = \frac{2\sigma}{\lambda_3^2} \frac{\partial w_3}{\partial x_3} + \lambda_3 \chi_3^* + \frac{q}{\lambda_3},$$

$$\chi_k^* = 2 \Big(\lambda_1 \chi_{k1} \frac{\partial w_1}{\partial x_1} + \lambda_2 \chi_{k2} \frac{\partial w_2}{\partial x_2} + \lambda_3 \chi_{k3} \frac{\partial w_3}{\partial x_3} \Big), \qquad k = 1, 2, 3.$$

We seek the components of the additional displacement vectors w_1, w_2, and w_3 and the linearized hydrostatic-pressure function q in the form

$$w_1 = W_1(x_3) \cos \gamma_1 x_1 \sin \gamma_2 x_2, \qquad w_2 = W_2(x_3) \sin \gamma_1 x_1 \cos \gamma_2 x_2,$$

$$w_3 = W_3(x_3) \sin \gamma_1 x_1 \sin \gamma_2 x_2, \qquad q = Q(x_3) \sin \gamma_1 x_1 \sin \gamma_2 x_2 \tag{2.13}$$

$$(\gamma_1 = \pi n/a, \qquad \gamma_2 = \pi m/b, \qquad m, n = 1, 2, 3),$$

which allows us to separate the variables x_1 and x_2 in the neutral-equilibrium equations (2.1) and (2.2) and boundary conditions (2.4) and reduce the stability problem to a linear homogeneous boundary-value problem for a system of ordinary differential equations. Functions (2.13) satisfy the following conditions at the plate edges (conditions of the first kind):

$$w_2|_{x_1=0,a} = 0, \qquad w_3|_{x_1=0,a} = 0, \qquad \Theta_{11}|_{x_1=0,a} = 0,$$

$$w_1|_{x_2=0,b} = 0, \qquad w_3|_{x_2=0,b} = 0, \qquad \Theta_{22}|_{x_2=0,b} = 0. \tag{2.14}$$

Taking into account (2.12) and (2.13), we write the linearized equilibrium equations (2.1) and (2.2) as

$$\frac{\lambda_1^2\chi_1-\sigma}{\lambda_1^2-\lambda_3^2}\,W_1''-\Big[\Big(\chi_1+\frac{\sigma}{\lambda_1^2}+2\lambda_1^2\chi_{11}\Big)\gamma_1^2+\frac{\lambda_1^2\chi_1-\lambda_2^2\chi_2}{\lambda_1^2-\lambda_2^2}\gamma_2^2\Big]W_1$$

$$-\Big(\frac{\sigma}{\lambda_1^2\lambda_2^2}+\frac{\chi_1-\chi_2}{\lambda_1^2-\lambda_2^2}+2\chi_{12}\Big)\lambda_1\lambda_2\gamma_1\gamma_2W_2+\Big(\frac{\lambda_1^2\chi_1-\sigma}{\lambda_1^2-\lambda_3^2}+2\lambda_1^2\chi_{13}\Big)\frac{\lambda_3\gamma_1}{\lambda_1}\,W_3'+\frac{\gamma_1}{\lambda_1}\,Q=0,$$

$$\frac{\lambda_2^2\chi_2-\sigma}{\lambda_2^2-\lambda_3^2}\,W_2''-\Big[\Big(\chi_2+\frac{\sigma}{\lambda_2^2}+2\lambda_2^2\chi_{22}\Big)\gamma_2^2+\frac{\lambda_1^2\chi_1-\lambda_2^2\chi_2}{\lambda_1^2-\lambda_2^2}\gamma_1^2\Big]W_2 \tag{2.15}$$

$$-\Big(\frac{\sigma}{\lambda_1^2\lambda_2^2}+\frac{\chi_1-\chi_2}{\lambda_1^2-\lambda_2^2}+2\chi_{12}\Big)\lambda_1\lambda_2\gamma_1\gamma_2W_1+\Big(\frac{\lambda_2^2\chi_2-\sigma}{\lambda_2^2-\lambda_3^2}+2\lambda_2^2\chi_{23}\Big)\frac{\lambda_3\gamma_2}{\lambda_2}\,W_3'+\frac{\gamma_2}{\lambda_2}\,Q=0,$$

$$-\Big(\frac{\lambda_1^2\chi_1-\sigma}{\lambda_1^2-\lambda_3^2}+2\lambda_1^2\chi_{13}\Big)\frac{\lambda_3\gamma_1}{\lambda_1}W_1'-\Big(\frac{\lambda_2^2\chi_2-\sigma}{\lambda_2^2-\lambda_3^2}+2\lambda_2^2\chi_{23}\Big)\frac{\lambda_3\gamma_2}{\lambda_2}\,W_2'$$

$$+\,2\Big(\frac{\sigma}{\lambda_3^2}+\lambda_3^2\chi_{33}\Big)W_3''-\Big(\frac{\lambda_1^2\chi_1-\sigma}{\lambda_1^2-\lambda_3^2}\,\gamma_1^2+\frac{\lambda_2^2\chi_2-\sigma}{\lambda_2^2-\lambda_3^2}\,\gamma_2^2\Big)W_3+\frac{1}{\lambda_3}Q'=0,$$

$$\frac{1}{\lambda_3}\,W_3'-\frac{\gamma_1}{\lambda_1}W_1-\frac{\gamma_2}{\lambda_2}\,W_2=0.$$

Using (2.12) and (2.13), we write boundary conditions (2.4) at the surfaces $x_3=\pm h/2$ in scalar form

$$\frac{\lambda_1^2\chi_1-\sigma}{\lambda_1^2-\lambda_3^2}\Big(W_1'+\frac{\lambda_3\gamma_1}{\lambda_1}\,W_3\Big)=0,\qquad \frac{\lambda_2^2\chi_2-\sigma}{\lambda_2^2-\lambda_3^2}\Big(W_2'+\frac{\lambda_3\gamma_2}{\lambda_2}\,W_3\Big)=0,$$

$$2(\sigma/\lambda_3^2+\lambda_3^2\chi_{33})W_3'-2\lambda_3(\lambda_1\gamma_1\chi_{13}W_1+\lambda_2\gamma_2\chi_{23}W_2)+q/\lambda_3=0,\qquad x_3=\pm h/2. \tag{2.16}$$

One can easily show that the boundary-value problem (2.15), (2.16) has two independent classes of solutions. One class is formed by solutions for which the deflection W_3 and the linearized hydrostatic pressure function Q are odd functions and the perturbation-vector components W_1 and W_2 are even functions. For solutions of the other class, in contrast, the components W_1 and W_2 are odd functions and W_3 and Q are even functions. Because of this property of the boundary-value problem, it suffices to study the stability question using only the upper half of the plate ($0\leqslant x_3\leqslant h/2$). The properties of the odd and even functions W_1, W_2, W_3, and Q imply the following boundary conditions for $x_3=0$:

$$W_1'|_{x_3=0}=W_2'|_{x_3=0}=W_3|_{x_3=0}=0 \tag{2.17a}$$

if W_3 is an odd function and

$$W_1|_{x_3=0}=W_2|_{x_3=0}=W_3'|_{x_3=0}=0 \tag{2.17b}$$

if W_3 is an even function.

It should be noted that the use of the representations

$$w_1=W_1(x_3)\sin\gamma_1x_1\cos\gamma_2x_2,\qquad w_2=W_2(x_3)\cos\gamma_1x_1\sin\gamma_2x_2,$$
$$w_3=-W_3(x_3)\cos\gamma_1x_1\cos\gamma_2x_2,\qquad q=-Q(x_3)\cos\gamma_1x_1\cos\gamma_2x_2; \tag{2.18}$$

$$w_1=W_1(x_3)\cos\gamma_1x_1\cos\gamma_2x_2,\qquad w_2=-W_2(x_3)\sin\gamma_1x_1\sin\gamma_2x_2,$$
$$w_3=W_3(x_3)\sin\gamma_1x_1\cos\gamma_2x_2,\qquad q=Q(x_3)\sin\gamma_1x_1\cos\gamma_2x_2; \tag{2.19}$$

$$w_1=-W_1(x_3)\sin\gamma_1x_1\sin\gamma_2x_2,\qquad w_2=W_2(x_3)\cos\gamma_1x_1\cos\gamma_2x_2,$$
$$w_3=W_3(x_3)\cos\gamma_1x_1\sin\gamma_2x_2,\qquad q=Q(x_3)\cos\gamma_1x_1\sin\gamma_2x_2 \tag{2.20}$$

also leads to the linear homogeneous boundary-value problem (2.15)–(2.17) for determining the functions W_1, W_2, W_3, and Q. At the plate edges, the solution of the form of (2.18) satisfies the conditions of the second kind (sliding clamped boundary conditions)

$$w_1|_{x_1=0,a} = 0, \qquad \Theta_{12}|_{x_1=0,a} = 0, \qquad \Theta_{13}|_{x_1=0,a} = 0,$$
$$w_2|_{x_2=0,b} = 0, \qquad \Theta_{21}|_{x_2=0,b} = 0, \qquad \Theta_{23}|_{x_2=0,b} = 0; \tag{2.21}$$

the solutions (2.19) satisfy the boundary conditions of the third kind

$$w_2|_{x_1=0,a} = 0, \qquad w_3|_{x_1=0,a} = 0, \qquad \Theta_{11}|_{x_1=0,a} = 0,$$
$$w_2|_{x_2=0,b} = 0, \qquad \Theta_{21}|_{x_2=0,b} = 0, \qquad \Theta_{23}|_{x_2=0,b} = 0;$$

the solutions (2.20) satisfy the boundary conditions the fourth kind

$$w_1|_{x_1=0,a} = 0, \qquad \Theta_{12}|_{x_1=0,a} = 0, \qquad \Theta_{13}|_{x_1=0,a} = 0,$$
$$w_1|_{x_2=0,b} = 0, \qquad w_3|_{x_2=0,b} = 0, \qquad \Theta_{22}|_{x_2=0,b} = 0.$$

3. Numerical Results and Discussion. The linearized boundary-value problem (2.15)–(2.17) is solved by the finite-difference method [3]. The effectiveness of the method was tested using the two-dimensional stability problem of a rectangular bar in tension with respect to small planar perturbations. This problem follows from problem (2.1)–(2.4) with allowance for representation (2.21) for $\gamma_2 = 0$ and $\lambda_2 = 1$. The numerical results were compared with the analytical solution of this problem [4]. The error is smaller than 0.01%.

The critical curves in the parameter plane (λ_1, λ_2) obtained from the solution of the boundary-value problem (2.15)–(2.17) possess some symmetry. In particular, the curve corresponding to the values $\gamma_1 = \mu$ and $\gamma_2 = \nu$ ($\mu \geqslant 0$ and $\nu \geqslant 0$ are some arbitrary values) and the curve corresponding to $\gamma_1 = \nu$ and $\gamma_2 = \mu$ are reflections in the line $\lambda_1 = \lambda_2$, and the critical curves obtained for $\gamma_1 = \gamma_2$ ($m = nb/a$) are symmetric about this line.

Stability analysis of a rectangular plate under biaxial tension was performed for the Biderman material model [5, 6], for which the specific strain energy is given by

$$W(G_1, G_2, G_3) = d_0(I_2 - 3) + d_1(I_1 - 3) + d_2(I_1 - 3)^2 + d_3(I_1 - 3)^3,$$
$$d_0 \geqslant 0, \qquad d_1 \geqslant 0, \qquad d_3 \geqslant 0, \qquad d_1 + d_3 \geqslant 0, \qquad 3d_2 + \sqrt{15 d_1 d_3} \geqslant 0$$

[$I_1 = \operatorname{tr} G$ and $I_2 = (1/2)(\operatorname{tr}^2 G - \operatorname{tr} G^2)$], a power-law material with the potential of the form [7, 8]

$$W(G_1, G_2, G_3) = d(I_1 - 3)^\beta, \qquad d > 0, \qquad \beta > 1/2,$$

and the Ogden material model with the potential of the form [9]

$$W(G_1, G_2, G_3) = d(\operatorname{tr} G^\beta - 3), \qquad d > 0, \qquad \beta \neq 0.$$

The numerical results given in the present paper were obtained for the following constants: $d_0 = 0$, $d_1 = 27$, $d_2 = -60$, and $d_3 = 80$ for the Biderman material, $d = 1$ and $\beta = 0.51$ for the power-law material, and $d = 1$ and $\beta = 0.1$ for the Ogden material. For these constants, all three materials satisfy the Hadamard condition [1, 6]. It should be borne in mind that if the energy function $W(G_1, G_2, G_3)$ is multiplied by an arbitrary positive constant, the critical ratio of the strain parameters λ_1 and λ_2 remains unchanged; it is therefore assumed for this problem that the specific energy is determined to within an arbitrary positive multiplier.

The parameter ξ is equal to the ratio of the plate thickness h to the length of the shorter side $\min\{a, b\}$ in the undeformed state of the body. In the present paper, the analysis is confined to plates with a normalized thickness $\xi \leqslant 1/2$.

Stability analysis was performed for tensile axial loads ($q_1 \geqslant 0$ and $q_2 \geqslant 0$). Using (1.7), we obtain the equations $q_1(\lambda_1, \lambda_2) = 0$ and $q_2(\lambda_1, \lambda_2) = 0$, which describe two curves in the parameter plane (λ_1, λ_2) that cut off the region of compressive loads. (In Figs. 1–3, the region of compressive loads is dashed.)

Figures 1 and 2 show the critical curves (the Biderman material model) in the plane of the parameters $\delta_1 = \lambda_1 - 1$ and $\delta_2 = \lambda_2 - 1$ for a square plate subject to the boundary conditions of the first kind (2.14) for the case where the deflection W_3 is an odd function. For each curve, the value of the parameter n ($\gamma_1 = \pi n/a$) to which

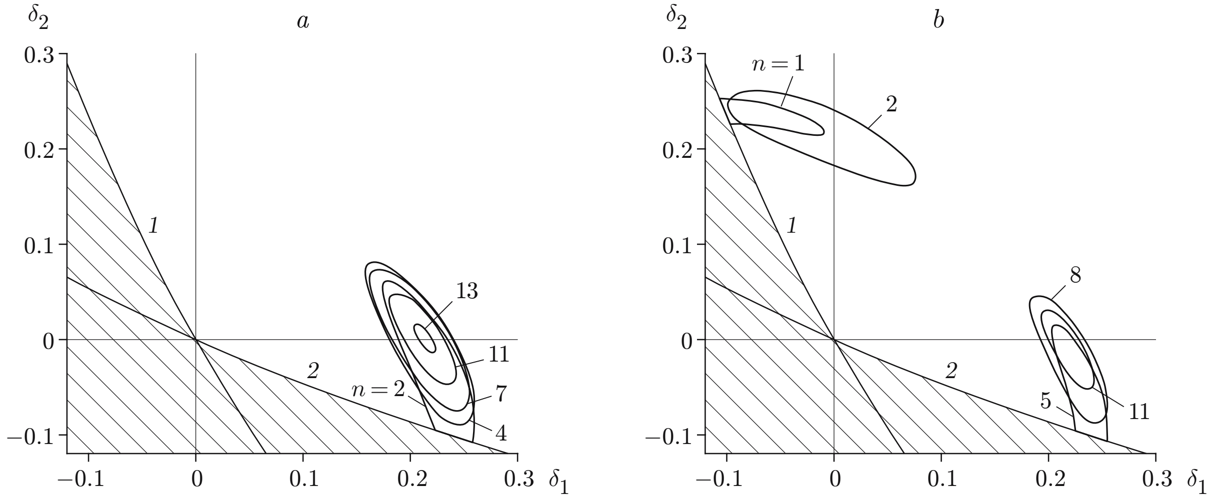

Fig. 1. Critical curves for the Biderman material ($\xi = 0.1$) for $m = 1$ (a) and 3 (b): curves 1 and 2 refer to $q_1 = 0$ and $q_2 = 0$, respectively.

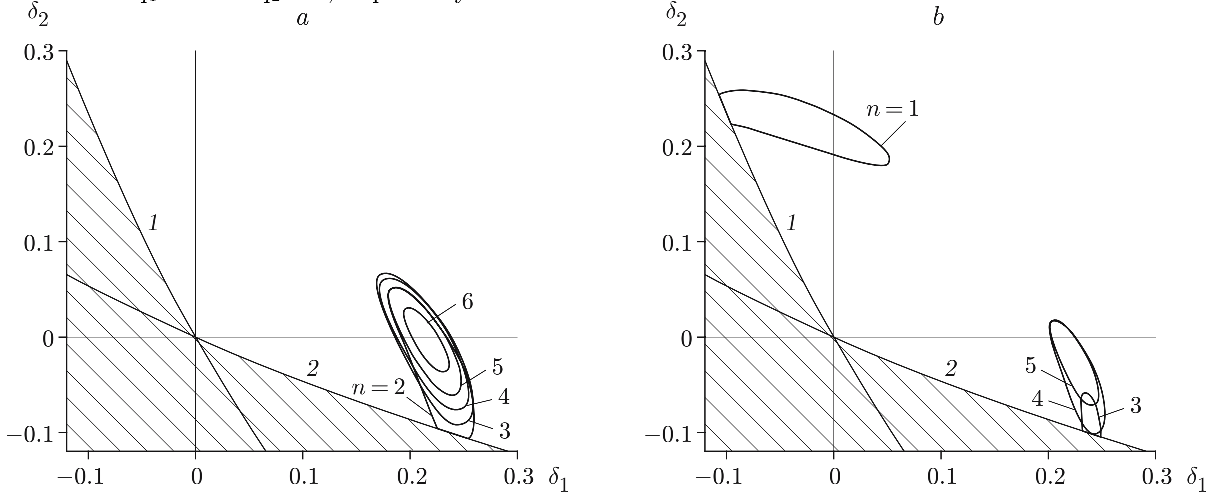

Fig. 2. Critical curves for the Biderman material ($\xi = 0.2$) for $m = 1$ (a) and 2 (b): curves 1 and 2 refer to $q_1 = 0$ and $q_2 = 0$, respectively.

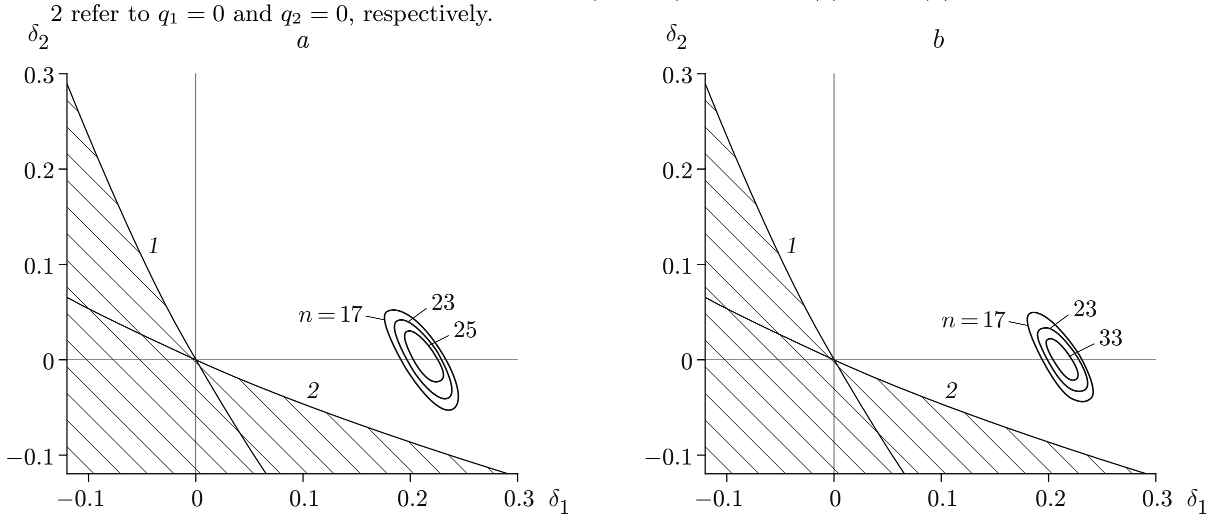

Fig. 3. Critical curves for the Biderman material for flexural buckling for $m = 1$ (a) and 3 (b): curves 1 and 2 refer to $q_1 = 0$ and $q_2 = 0$, respectively.

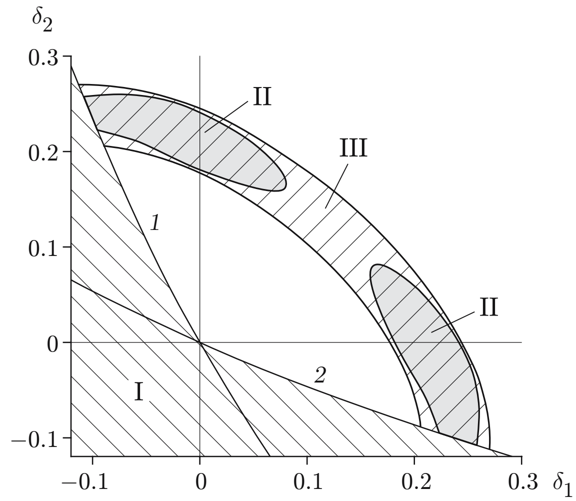

Fig. 4. Stability region for the Biderman material ($\xi = 0.1$): I is the region of compressive loads, II is the instability region, and III is the region in which the convexity condition (4.1) is violated; curves 1 and 2 refer to $q_1 = 0$ and $q_2 = 0$, respectively.

the curve corresponds is indicated. One can see from Figs. 1 and 2 that as the plate thickness decreases, the curves are spaced more closely to each other. This effect is observed for all the materials considered.

For the case of flexural buckling (the deflection W_3 is an even function), the critical curves are shown in Fig. 3 for $\xi = 0.1$ and $a = b$. According to the results obtained for thin plates ($\xi \leqslant 0.1$), lower-order flexural modes do not occur. Moreover, the existence region of flexural buckling modes is embedded in the existence region of solutions of the boundary-value problem (2.15)–(2.17) for which W_3 is an odd functions. Consequently, to construct the stability region for biaxial tension of the plate, it suffices to study only the buckling modes for which W_3 is an odd function; in this case, one need not find unstable flexural modes. Results obtained for the other materials support this conclusion.

Figure 4 shows the stability region for a square plate (the Biderman material model) with boundary conditions of the first kind. As can be seen from Fig. 4, the stability region is symmetric about the line $\lambda_1 = \lambda_2$, which follows from the above-mentioned symmetry property of the solution of the boundary-value problem (2.15)–(2.17). Calculations show that for thin plates ($\xi \leqslant 0.1$), the size and shape of the stability region depend only slightly on the plate thickness and the aspect ratio.

Figure 5 shows the stability regions for $\xi = 0.1$ and $a = b$ for the Ogden material and power-law material models.

If boundary conditions of the second kind (a sliding clamped edge) are specified at the plate edges, the linearized boundary-value problem is solved, as indicated above, using representations (2.18) [if the conditions of the third or fourth kinds are specified, representations (2.19) or (2.20), respectively, are used]. In this case, the spectrum of the critical curves for $m, n = 1, 2, 3, \ldots$ is the same as for a plate subject to boundary conditions of the first kind. At the same time, solutions of the form of (2.18)–(2.20) admit, in contrast to (2.13), the occurrence of planar instability modes (when one component of the perturbation vector $\boldsymbol{w}$ vanishes). To obtain the critical curves corresponding to these buckling modes, one should solve the boundary-value problem (2.15)–(2.17) for $m = 0$, $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$, $n = 0$. It was found from the calculations results that the stability region for a thin plate subject to the boundary conditions of the first kind differs only slightly from that for the plate subject to the sliding clamped boundary conditions or conditions of the third and fourth kinds. The properties of the stability region of thin plates ($\xi \leqslant 0.1$) mentioned above were found for all the material models considered in the present paper.

4. Stability Criterion. We consider the total potential strain energy of the plate Π as a function the parameters λ_1 and λ_2:

$$\Pi(\lambda_1, \lambda_2) = \iiint\limits_v W(G_1, G_2, G_3)\, dv.$$

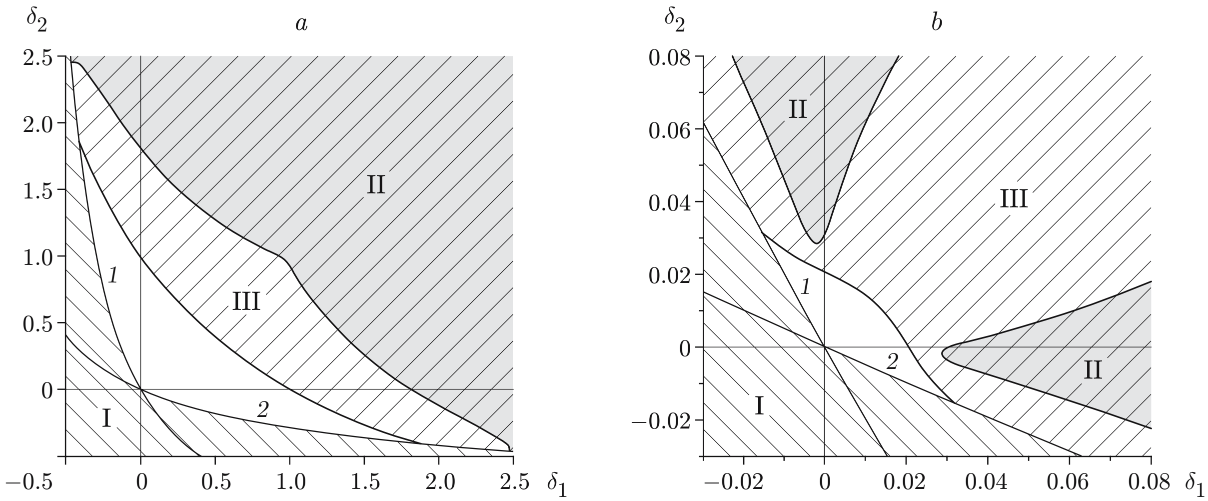

Fig. 5. Stability regions for the Ogden material (a) and power-law material (b) (notation the same as in Fig. 4).

For the function $\Pi(\lambda_1, \lambda_2)$, the rigorous convexity condition is given by

$$\frac{\partial^2 \Pi}{\partial \lambda_1^2} > 0, \qquad \frac{\partial^2 \Pi}{\partial \lambda_1^2}\frac{\partial^2 \Pi}{\partial \lambda_2^2} - \Big(\frac{\partial^2 \Pi}{\partial \lambda_1\, \partial \lambda_2}\Big)^2 > 0. \tag{4.1}$$

In view of relations (1.7) and the fact that the initial (subcritical) stress-strain state is uniform, this condition can be written in the form of the Drucker postulate [10]

$$dq_1\, d\lambda_1 + dq_2\, d\lambda_2 > 0.$$

The last condition implies the inequalities

$$\frac{\partial q_1}{\partial \lambda_1} > 0, \qquad \frac{\partial q_1}{\partial \lambda_1}\frac{\partial q_2}{\partial \lambda_2} - \frac{\partial q_1}{\partial \lambda_2}\frac{\partial q_2}{\partial \lambda_1} > 0.$$

In the case of the two-dimensional stability problem ($\lambda_2 = 1$) of an elastic rectilinear bar in tension, there exists a theorem (which is proved in [4] for an arbitrary incompressible material) that states that no bifurcation of the equilibrium of the bar in tension occurs at the ascending part of the stress–strain diagram $q_1(\lambda_1)$. In the plane problem, because $q_1(\lambda_1) = (abh)^{-1}\partial\Pi(\lambda_1)/\partial\lambda_1$, it follows that the inequality $\partial q_1/\partial\lambda_1 = (abh)^{-1}\partial^2\Pi/\partial\lambda_1^2 > 0$, i.e., the convexity condition for the energy function $\Pi(\lambda_1)$, is a sufficient stability condition for the equilibrium of the bar in tension, which is supported by numerical results obtained in the present study. Previously, for the stability problem of a cylinder subjected to combined tension and torsion, it has been shown that the accuracy of the convex potential energy criterion provides reasonable accuracy for practical use [11].

In the case of a rectangular plate under biaxial tension, it is of interest to compare the strain stability region and the convexity region of the energy $\Pi(\lambda_1, \lambda_2)$. For the plate subject to boundary conditions of the first kind, these regions are shown in Figs. 4 and 5 for the Biderman, Ogden, and power-law materials. One can see that in the range where the convexity condition of the energy Π holds, instability does not occur (regardless of the plate dimensions). The regions in which the convexity and instability conditions are violated for thin plates ($\xi \leqslant 0.1$) differ from those shown in Figs. 4 and 5 only slightly, but as the thickness increases, the difference in the size and shape of these regions becomes more pronounced since the instability region is diminished with increasing thickness whereas the size and shape of the convexity region do not depend on the plate dimensions.

Comparison results of the stability and convexity regions of the energy for other types of boundary conditions [conditions of the second kind (2.18), third kind (2.19), or fourth kind (2.20)] agree qualitatively with the results obtained for the plate subject to the conditions of the first kind.

It should be noted that a comparison of the stability and convexity regions of the energy is meaningful only for tensile boundary loads ($q_1 \geqslant 0$ and $q_2 \geqslant 0$). Indeed, as calculations show, if only one of the loads q_1 and q_2 is compressive, a loss of stability is possible in the convexity region of the energy $\Pi(\lambda_1, \lambda_2)$. For example, a square plate ($\xi = 0.1$; the Biderman material model) with boundary conditions of the first kind subjected to biaxial tension by loads of equal intensity ($q_1 = q_2 < 0$) loses stability for a strain $\lambda_1 = \lambda_2 = 0.9891$, whereas the convexity condition (4.1) always holds for $\lambda_1 = \lambda_2 < 1$.

Thus, the convexity property of the total potential energy $\Pi(\lambda_1, \lambda_2)$ is a sufficient stability condition for tensile boundary loads and can be used as a stability criterion in solving practical problems of rectangular plates under biaxial tension. A drawback of this criterion is that the convexity region of the total potential energy and the stability region obtained by solving the linearized boundary-value problem (2.15)–(2.17) can differ substantially for some material models.

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